Theoretische Physik

On Quantum Chaos, Stochastic Webs and Localization in a Quantum Mechanical Kick System

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Introduction

Quantum chaos — is there any?

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Directions in Chaos Vol. 2 [For88]

On Quantum Chaos

Quantum chaos is a controversial subject [SZ94]. On the basis of the correspondence principle one might be tempted to take quantum chaos for granted, because classical mechanics is supposed to be a special limiting case of quantum mechanics, and clearly there are classical dynamical systems that are known to exhibit chaos — cf. for example [AFH95, KH95a] and the comprehensive lists of references therein.

But this approach is problematic: while stating that in the limit $\hbar = 0$ the quantum dynamics transforms into the classical dynamics of the system, the correspondence principle does not say *how* this quantum-classical transition comes about. There is no well-defined algorithm that maps the two dynamical theories onto each other; the semiclassical limit $\hbar \to 0$ of quantum mechanics is mathematically singular [Ber01]. With such an algorithm, given a classically chaotic system, it would be possible to uniquely identify those properties of the corresponding quantum system that mark its quantum chaoticity, based on the well-established concepts of classical chaos theory: LIAPUNOV exponents, (fractal) dimensions, entropies, etc. Conversely, since there is no such algorithm and the concepts of chaotic dynamics originate in the theory of classical mechanics, until today nobody can define in a straightforward way what makes a quantum system
2 chaotic and thus what quantum chaos really is. With such a definition it would

be possible to identify a quantum system as chaotic without having to discuss its classical mechanics beforehand.

Another aspect of the same problem is the fact that quantum mechanical phase space is quantized into cells of finite size, given by $\hbar$. As a consequence, the concept of sensitive dependence on initial conditions — an essential ingredient of the theory of classical chaos — cannot be applied to quantum mechanics in a straightforward way. What is more, if the (region of) phase space under consideration is of finite volume $\mathcal{A}$ then the corresponding quantum dynamics effectively evolves on $\mathcal{A}/\hbar$ different phase space cells only, which makes the HILBERT space effectively finite-dimensional, brings about quasiperiodic dynamics and thus excludes chaos in the classical sense [KW96]. This is to be contrasted with the continuum of phase space points in any nontrivial classical phase space and illustrates again the fundamental differences between the two dynamical theories.

Note that the above reasoning leading to the absence of classical chaos in quantum mechanics is also based on the *linearity* of the SCHRÖDINGER equation; the canonical equations of motion of classical mechanics, on the other hand, are in general *nonlinear*. BERRY considers this distinction to be "the ultimate reason for the absence of quantum chaos" [Ber89].

To give an example, since the days of POINCARÉ the three-body problem [Bru94] is a paradigm of classically chaotic nonlinear dynamics — see [Klu97, Hip97] for some recent studies of classical molecular three-body systems and [Con02] for applications in astronomy. However, its quantum counterparts, for example the H_2^+ ion and the He atom, can be analyzed with relative ease using conventional quantum techniques [Nol02], their eigenvalues and eigenfunctions are obtained accurately, and from this point of view it is difficult to see why these apparently well-behaved quantum three-body problems should be examples of quantum chaos.

Nevertheless, motivated by random matrix theory, one also has the BOHIGAS-GIANNONI-SCHMIT conjecture [BGS84] which claims that *there are* manifestations of classical chaos in quantum energy level spacing distributions: classical integrability is associated with POISSON distributions and level clustering, whereas for (hard) classical chaos WIGNER distributions and level repulsion are expected. The conjecture is yet unproven, but there is much evidence for its validity [Stö99]. — Similarly, there is the general conjecture that classically chaotic systems with diffusive dynamics are characterized by quantum suppression of diffusion [CCIF79, Ber89]: another quantum fingerprint of classical chaos.

A pragmatic way out of this dilemma is to adopt a working definition of quantum chaos that avoids the correspondence principle altogether, and to define *quantum chaos* as the study of quantum systems whose classical counterparts exhibit classical chaos [CGS93]. This definition — often also termed *quantum chaology* [Ber89], trying to avoid the impression that the quantum dynamics might be chaotic per se — appears to be accepted by more and more researchers in the field (see for example [GVZJ91, CC95, Stö99, Haa01]). Note that this definition makes no ad hoc statement about the actual properties of the quantum systems, other than their classical chaoticity.

With the above definition, quantum chaos integrates smoothly into the development of the theory of nonlinear dynamics, or chaos theory, as it has taken place since the middle of the last century. Using both the increasing theoretical understanding of the dynamics and computers which constantly became more powerful, a lot of knowledge about classical dynamical systems has been accumulated. The natural next step is to look for the quantum manifestations of classical chaos. With the present study I want to make a contribution to this development by investigating the quantum mechanics of a system the chaotic classical mechanics of which has been studied in some detail in the past: the *kicked harmonic oscillator*.

Among the classically chaotic systems, those that exhibit *hard chaos* take an exceptional role. Hard chaotic systems are characterized by complete absence of regular regions in classical phase space: all orbits, including the periodic orbits, are unstable. Once all periodic orbits of such a system are known, the quantum mechanics of the system can be studied in the semiclassical approximation $\hbar \approx 0$ using, for example, GUTZWILLER's trace formulas [Gut90, Gut91]. This theory is especially successful for bound dynamics [BB97], but has been extended to scattering dynamics as well [Wir99, CAM$^+$03].

Billiards represent a frequently studied class of model systems for semiclassical quantization. Inside a billiard, the dynamics is free, and therefore easy to solve classically and quantum mechanically. The billiard boundaries then account for the nontrivial part of the dynamics; in particular, defocusing boundaries can be chosen such that hard chaotic classical dynamics with a complete symbolic dynamics is obtained, and the methods of semiclassical quantization can be applied [Hor93, Jun97, Hau00]. Two-dimensional quantum billiard systems have the striking additional advantage of being accessible to direct experimental verification of the numerical and analytical results: these systems can be modelled by microwave resonators, also known as microwave billiards, which allow to measure the wave function of the system using a macroscopic experimental setup; this method is based on the fact that for two-dimensional systems, both the SCHRÖDINGER equation for the quantum billiard and the MAXWELL equations for the microwave resonator give rise to the same HELMHOLTZ equation with essentially identical boundary conditions [Stö99].

The generic situation in nonlinear Hamiltonian dynamical systems, on the other hand, is *soft* or *weak chaos*. Here, the phase portrait is mixed and consists of regions of regular and irregular or chaotic dynamics at the same time. Often such systems can be analyzed from a perturbation point of view, starting from a completely integrable system to which an increasingly strong perturbation is added, rendering the system nonintegrable and leading to growing phase space regions of chaotic dynamics.

The kicked harmonic oscillator is a typical representative of this class of systems, the integrable component being the well-known harmonic oscillator to which an impulsive time-periodic forcing is added. The kicked harmonic oscillator is *un*typical, however, in that respect that it can generate a very special kind of classical phase portraits, namely *stochastic webs*. These extended, periodic structures in phase space contain both bounded cells of regular dynamics and infinitely long channels of chaotic, or stochastic, motion. Due to the degeneracy of the harmonic oscillator, its classical dynamics cannot be analyzed in terms of the elegant standard theory of perturbed integrable systems, KAM theory [LL92, JS98]. Rather, a theory of its own had to be developed in order to understand the stochastic webs that are generated by this system [ZSUC91].

In contrast to classically hard chaotic systems, the quantum mechanics of systems exhibiting weak chaos is much less understood from a general point of view. The methods of semiclassical quantization via periodic orbit theory, as mentioned above, cannot be applied in a straightforward way. Therefore, a significant part of the quantum investigations of weakly chaotic systems have been performed using quantum mechanical methods without $\hbar$-related approximations, allowing to consider both the full quantum regime and the semiclassical case [Zas85, Hei92, CC95, BR97] (and references therein). The same approach is followed in the present study.

Beyond the lack of a suitable semiclassical theory, studying the quantum dynamics of the kicked harmonic oscillator is further complicated by the quantum fingerprints of its diffusive dynamics in the channels of the classical webs: in cases of resonance between the harmonic oscillator's eigenfrequency and the frequency of the perturbation, the evolving quantum states can extend quite rapidly into large areas in phase space, which greatly increases the numerical effort needed to study the system. It is interesting to note how this numerical complexity of the system has been assessed by some prominent researchers in this field:

> "The numerical investigation of this model [the quantum kicked harmonic oscillator] is much more difficult than in the kicked Harper case, since the dynamics leads to spreading in the whole (p, x)-plane... This seems to be the reason why there were practically no attempts to investigate the quantum dynamics of [this] model" [SS92].

While this remark is slightly outdated by now, it is still true that the kicked harmonic oscillator is a seldom studied object with respect to quantum chaos. To put that into perspective, the quantum dynamics of the paradigmatic example of nonlinear Hamiltonian kick dynamics, the *kicked rotor* which gives rise to the *standard map*, has been studied in hundreds of publications, while for the kicked harmonic oscillator there are still only a few in comparison. Among other — more theoretical — reasons, this is mainly due to the fact that it is *much* easier to numerically model the kicked rotor than the kicked harmonic oscillator; this might become clearer in chapter 5, where both systems are studied along parallel lines.

Outline of this Study

Chapter 1 is the exposition of the principal model system used in the present study, the kicked harmonic oscillator. The Hamiltonian of the model system is introduced and some sample applications to physical systems are discussed. After scaling, the POINCARÉ map describing the classical dynamics over one period of the excitation is derived. In cases of resonance, this *web map* is then used to numerically generate classical stochastic webs in phase space and to explain some of their most important properties, such as the topology and the symmetries of the webs, and the diffusive dynamics in the channels. In cases of nonresonance, the discrete map does not give rise to web-like structures and the dynamics is typically diffusive. — While this chapter does not contain much original material, it serves two important purposes. First, in comparison with the existing literature on the subject it gives a more readable account of — some important aspects of — the theory of stochastic webs. Second, it provides the basic classical results with which the quantum results of the following chapters are to be compared.

In chapter 2 I describe the quantum formulation of the problem. FLOQUET theory is employed to derive the *quantum map* which is the quantum analogue of the classical POINCARÉ map. The similarities and the fundamental differences of these two mappings are discussed.

Studying the quantum analogue of stochastic webs requires the iteration of the quantum map for a very large number of times. This cannot be done analytically; it can only be accomplished using numerical means. Since the numerical effort to be spent for a single iteration of the quantum map is *much* larger than for one iteration of the classical POINCARÉ map, it is important to select the most efficient algorithm that is available. In chapter 3 I present and compare three numerical methods that can be used to implement the quantum map on a computer. It turns out that representing the FLOQUET operator in the eigenbasis of the (unkicked) harmonic oscillator is better suited for the present study than using conventional finite differences methods.

Chapters 4 and 5 contain the core results of this study: for several parameter combinations, I iterate the quantum map very often, compare the resulting sequences of quantum states with the corresponding classical dynamics, and give analytical explanations for the observations. For this comparison, a technique is needed that is reviewed in appendix A: the theory of *quantum phase space distribution functions* can be used to define a quantum analogue of classical phase space; the quantum states are then described equivalently in terms of distribution functions in this quantum phase space that take the role of the classical LIOUVILLE distribution. In this way a direct comparison of the classical and quantum results becomes possible.

In chapter 4, the quantum dynamics in the resonance cases is studied numerically, with the result that there exist *quantum stochastic webs* if and only if there are classical stochastic webs. The quantum webs resemble their classical counterparts as closely as allowed by the value of $\hbar$, and generically the dynamics in the channels of the webs is diffusive, as in the classical case. In other words, in the quantum webs the dynamics is as classical as can be expected from a quantum wave packet. These numerical findings are then explained analytically using an argument that relies on exploiting the symmetries of the FLOQUET operator and on constructing groups of mutually commuting translation operators in the phase plane that also commute with the FLOQUET operator [BR95].

The complementary case of nonresonance is dealt with in chapter 5. It turns out that in this case the dynamics is similar — in a well-defined way — to the dynamics of the quantum kicked rotor, which is known for some time already to exhibit quantum suppression of diffusion, or *quantum localization* [CCIF79, FGP82]. In chapter 5 I show both numerically and analytically that the same is true for the model system considered here: the nonresonant quantum kicked harmonic oscillator is ANDERSON-localized.

Finally, appendix B contains some technical material needed for the proof of localization in chapter 5, and appendix C is a collection of sample quantum phase portraits of the dynamics of the kicked harmonic oscillator, both in cases of resonance and nonresonance.